MATH 1010K 2017-18

University Mathematics

Tutorial Notes I

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Question

Q1. State whether the following sequence converges.

If no, just answer "the sequence is not convergent" without giving any justification.

If yes, find the limit.

(a) $a_n = \dfrac{n^3 + 7n^2 + 8n - 1}{2n^3 - 6n^2 + 5}$,

(b) $a_n = \dfrac{n^4 + 5n + 2}{n^3 + 2n^2}$,

(c) $a_n = \sqrt[3]{n+5} - \sqrt[3]{n}$,

(d) $a_n = \sin \dfrac{n\pi}{2}$.

Q2. Given that

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1, \quad \lim_{n\to\infty} n \sin\frac{1}{n} = 1,$$

evaluate

$$\lim_{n\to\infty} n^{\frac{2+n}{n}} \sin\frac{1}{n}.$$

Q3. Let $a_1 = 5$, $a_{n+1} = \sqrt{1 + a_n}$ for any $n \in \mathbb{N}$.

First, show that $a_n > 0$ for any $n \in \mathbb{N}$ and then *assume* a_n converges, find its limit.

Q4. Let $a_n = \dfrac{1}{3^n} + 1$, is a_n

(a) monotone?

(b) bounded above?

(c) bounded below?

Q5. Using the Sandwich theorem to evaluate the following limit

$$\lim_{n\to\infty} \frac{\sqrt{n^2+1} - \cos\sqrt{n^2+1}}{\sqrt{n^2+1} + \cos\sqrt{n^2+1}}$$

Challenging Question Suppose $a_0 = \dfrac{10}{3}$, $a_k = a_{k-1}^2 - 2$ for any $k \in \mathbb{N}$.

(a) Show that $a_k = 3^{2^k} + 3^{-2^k}$ for all $k \in \mathbb{N}$ and $k = 0$.

(b) Show that $\displaystyle\prod_{k=0}^{n} a_k = \frac{3^{2^{n+1}} - 3^{-2^{n+1}}}{3 - 3^{-1}}$ for all $n \in \mathbb{N}$ and $n = 0$.

(c) Show that $\displaystyle\prod_{k=0}^{n} (a_k - 1) = \frac{3^{2^{n+1}} + 3^{-2^{n+1}} + 1}{3 + 3^{-1} + 1}$ for all $n \in \mathbb{N}$ and $n = 0$.

(d) Compute $\displaystyle\lim_{n\to\infty} \prod_{k=0}^{n} \left(1 - \frac{1}{a_k}\right)$.

Answer

A1(a). $\lim_{n\to\infty} \frac{n^3+7n^2+8n-1}{2n^3-6n^2+5} = \lim_{n\to\infty} \frac{1+\frac{7}{n}+\frac{8}{n^2}-\frac{1}{n^3}}{2-\frac{6}{n}+\frac{5}{n^2}} = \frac{1+0+0-0}{2-0+0} = \frac{1}{2}.$

A1(b). Since the degree of the numerator $= 4 > 3 =$ the degree of denominator,

the sequence is not convergent.

A1(c). Using the formula $(a-b)^3 = (a-b)(a^2+ab+b^2)$, we have

$$\lim_{n\to\infty} \left(\sqrt[3]{n+5} - \sqrt[3]{n}\right) = \lim_{n\to\infty} \frac{n+5-n}{(n+5)^{\frac{2}{3}} + \left(n(n+5)\right)^{\frac{1}{3}} + n^{\frac{2}{3}}} = \lim_{n\to\infty} \frac{\frac{5}{n^{\frac{2}{3}}}}{\left(1+\frac{5}{n}\right)^{\frac{2}{3}} + \left(1\times\left(1+\frac{5}{n}\right)\right)^{\frac{1}{3}} + 1}$$

$$= \frac{0}{1+1+1} = 0.$$

A1(d). Note that $a_n = \begin{cases} 0, & \text{if } n = 2,4,6,... \\ 1, & \text{if } n = 1,5,9,... \\ -1, & \text{if } n = 3,7,11,... \end{cases}$, even when n is large, a_n still oscillate on $-1, 0, 1$,

the difference of a_n, a_m (for any n, m are very large) is NOT small,

so the sequence is not convergent.

A2. $\lim_{n\to\infty} n^{\frac{2+n}{n}} \sin\frac{1}{n} = \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)\left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)\left(\lim_{n\to\infty} n\sin\frac{1}{n}\right) = (1)(1)(1) = 1.$

A3. Let $P(n)$ be te statement that $a_n > 0$.

By $a_1 = 5 > 0$, $P(1)$ is true.

Assume $P(k)$ is true for some $k \in \mathbb{N}$, i.e. $a_k > 0$,

Then $1 + a_k > 0$, so $a_{k+1} = \sqrt{1+a_k} > 0$, so $P(k+1)$ also true.

By first principal of mathematical induction, $P(n)$ is true for any $n \in \mathbb{N}$.

i.e. $a_n > 0$ for any $n \in \mathbb{N}$.

Assume a_n converge, let $a = \lim_{n\to\infty} a_n$, then we have

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{1+a_n} \\ a &= \sqrt{1+a} \\ a^2 &= 1+a \\ a &= \frac{1\pm\sqrt{5}}{2}. \end{aligned}$$

Since $a_n > 0$ for any $n \in \mathbb{N}$, we have $a \geq 0$,

the value that $a = \frac{1-\sqrt{5}}{2} < 0$ need to be rejected.

Hence, $\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}$.

A4(a). Note that

$$\begin{aligned} 1 &< 3, \\ 3^n &< 3^{n+1}, \\ \frac{1}{3^n} &> \frac{1}{3^{n+1}}, \\ a_n = \frac{1}{3^n} + 1 &> \frac{1}{3^{n+1}} + 1 = a_{n+1}, \end{aligned}$$

is true for any $n \in \mathbb{N}$, so a_n is decreasing.

A4(b). Note that $1 > \frac{1}{3^n}$ for any $n \in \mathbb{N}$, so $2 > a_n$ for any $n \in \mathbb{N}$.

Then a_n is bounded above with an upper bound 2.

A4(c). Note that $\frac{1}{3^n} > 0$ for any $n \in \mathbb{N}$, so $a_n > 1$ for any $n \in \mathbb{N}$.

Then a_n is bounded below with an lower bound 1.

A5. Note that

$$\frac{\sqrt{n^2+1}-1}{\sqrt{n^2+1}+1} \le \frac{\sqrt{n^2+1} - \cos\sqrt{n^2+1}}{\sqrt{n^2+1} + \cos\sqrt{n^2+1}} \le \frac{\sqrt{n^2+1}+1}{\sqrt{n^2+1}-1}$$

true for any $n \in \mathbb{N}$. Also note that

$$\lim_{n\to\infty} \frac{\sqrt{n^2+1}-1}{\sqrt{n^2+1}+1} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n^2}} - \frac{1}{n}}{\sqrt{1+\frac{1}{n^2}} + \frac{1}{n}} = \frac{\sqrt{1+0}-0}{\sqrt{1+0}+0} = 1.$$

Similarly, we can have $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^2+1}+1}{\sqrt{n^2+1}-1} = 1.$

By Sandwich Theorem, $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^2+1} - \cos\sqrt{n^2+1}}{\sqrt{n^2+1} + \cos\sqrt{n^2+1}}$ exists and equal to 1.

Challenging Question (a). Let $P(k)$ be the statement that $a_k = 3^{2^k} + 3^{-2^k}$.

Note that $\dfrac{10}{3} = a_0 = 3^{2^0} + 3^{-2^0}$, so $P(0)$ is true.

Assume $P(l)$ is true for some $l \in \mathbb{N}$ or $l = 0$, then

$$a_{l+1} = a_l^2 - 2 = \left(3^{2^l} + 3^{-2^l}\right)^2 - 2 = 3^{2^l \cdot 2} + 2 + 3^{-2^l \cdot 2} - 2 = 3^{2^{l+1}} + 3^{-2^{l+1}}.$$

Hence, $P(l+1)$ also true.

By first principal of mathematical induction, $P(k)$ is true for any $k \in \mathbb{N}$ and $k = 0$.

i.e. $a_k = 3^{2^k} + 3^{-2^k}$ for any $k \in \mathbb{N}$ and $k = 0$.

Challenging Question (b). Let $Q(n)$ be the statement that $\displaystyle\prod_{k=0}^{n} a_k = \frac{3^{2^{n+1}} - 3^{-2^{n+1}}}{3 - 3^{-1}}$.

Note that $\displaystyle\prod_{k=0}^{0} a_k = a_0 = \frac{10}{3} = \frac{3^2 - 3^{-2}}{3 - 3^{-1}}$, so $Q(0)$ is true.

Assume $Q(m)$ is true for some $m \in \mathbb{N}$ or $m = 0$, then (Using $(a+b)(a-b) = a^2 - b^2$)

$$\prod_{k=0}^{m+1} a_k = \left(3^{2^{m+1}} + 3^{-2^{m+1}}\right)\left(\frac{3^{2^{m+1}} - 3^{-2^{m+1}}}{3 - 3^{-1}}\right) = \frac{3^{2^{m+2}} - 3^{-2^{m+2}}}{3 - 3^{-1}}.$$

Hence, $Q(m+1)$ also true.

By first principal of mathematical induction, $Q(n)$ is true for any $n \in \mathbb{N}$ and $k = 0$.

i.e. $\displaystyle\prod_{k=0}^{n} a_k = \frac{3^{2^{n+1}} - 3^{-2^{n+1}}}{3 - 3^{-1}}$ for any $n \in \mathbb{N}$ and $n = 0$.

Challenging Question (c). Let $R(n)$ be the statement that $\displaystyle\prod_{k=0}^{n}(a_k - 1) = \frac{3^{2^{n+1}} + 3^{-2^{n+1}} + 1}{3 + 3^{-1} + 1}$.

Note that $\displaystyle\prod_{k=0}^{0}(a_k - 1) = (a_0 - 1) = \frac{7}{3} = \frac{3^2 + 3^{-2}}{3 + 3^{-1} + 1}$, so $R(1)$ is true.

Assume $R(m)$ is true for some $m \in \mathbb{N}$ or $m = 0$, then (Using $(a+b)(a-b) = a^2 - b^2$)

$$\begin{aligned}\prod_{k=0}^{n+1}(a_k - 1) &= \left(3^{2^{m+1}} + 3^{-2^{m+1}} - 1\right)\left(\frac{3^{2^{m+1}} + 3^{-2^{m+1}} + 1}{3 + 3^{-1} + 1}\right)\\ &= \frac{\left(3^{2^{m+1}} + 3^{-2^{m+1}}\right)^2 - 1}{3 + 3^{-1} + 1}\\ &= \frac{3^{2^{m+2}} + 3^{-2^{m+2}} + 1}{3 + 3^{-1} + 1}\end{aligned}$$

Hence, $R(m+1)$ also true.

By first principal of mathematical induction, $R(n)$ is true for any $n \in \mathbb{N}$ and $k = 0$.

i.e. $\displaystyle\prod_{k=0}^{n}(a_k - 1) = \frac{3^{2^{n+1}} + 3^{-2^{n+1}} + 1}{3 + 3^{-1} + 1}$ for any $n \in \mathbb{N}$ and $n = 0$.

Challenging Question (d). Using (b),(c),

$$\begin{aligned}\lim_{n\to\infty}\prod_{k=0}^{n}\left(1 - \frac{1}{a_k}\right) &= \lim_{n\to\infty}\frac{\prod_{k=0}^{n}(a_k - 1)}{\prod_{k=0}^{n} a_k}\\ &= \frac{3 - 3^{-1}}{3 + 3^{-1} + 1}\lim_{n\to\infty}\frac{3^{2^{n+1}} + 3^{-2^{n+1}} + 1}{3^{2^{n+1}} - 3^{-2^{n+1}}}\\ &= \frac{8}{13}\lim_{n\to\infty}\frac{1 + 3^{-2^{n+2}} + 3^{-2^{n+1}}}{1 - 3^{-2^{n+2}}}\\ &= \frac{8}{13}.\end{aligned}$$